# A note on the Markoff condition and central words

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**Abstract**

We define *Markoff words* as certain factors appearing in bi-infinite words satisfying the *Markoff condition*. We prove that these words coincide with *central words*, yielding a new characterization of *Christoffel words*.

*Key words:* combinatorial problems, Markoff condition, balanced words, central words, Christoffel words, palindromes.
*1991 MSC:* 68R15.

## 1. Introduction

In studying the minima of certain binary quadratic forms $AX^2+2BXY+CY^2$, Markoff [8,9] introduced a necessary condition that a bi-infinite word $\boldsymbol{s}$ must satisfy in order that it represent the continued fraction expansions of the two roots of $AX^2 + 2BX+C$. Over an alphabet $\{a,b\}$, his condition essentially states that each factor $x\widetilde{m}abmy$ occurring in $\boldsymbol{s}$, where $\widetilde{m}$ is the word $m$ read in reverse and $\{x,y\} = \{a,b\}$, has the property that $x = b$ and $y = a$. We call such words $m$ *Markoff words* in what follows. See Definition 2.

From [11] (see also [2, pg. 30]), it is known that the bi-infinite words satisfying the Markoff condition are precisely the *balanced words* of Morse and Hedlund [10]. After the work of A. de Luca [3,4], we know that palindromes now play a 'central' role in the study of such words. Here, we establish the following new characterization of a particular family of palindromes called *central words*.

**Theorem 1** *A word is a Markoff word if and only if it is a central word.*

Central words hold a special place in the rich theory of *Sturmian words* (e.g., see [7, Chapter 2]). For instance, it follows from the work of de Luca and Mignosi [4,5] that central words coincide with the palindromic prefixes of standard Sturmian words.

As an immediate consequence of Theorem 1, we obtain a new characterization of *Christoffel words* in Corollary 7. Since the Markoff condition is relatively unknown, we discuss it and its relationship to Christoffel words at greater length in Section 5.

## 2. The Markoff condition

Fix an alphabet $\{a,b\}$. A finite sequence $a_1$, $a_2, \ldots, a_n$ of elements from $\{a,b\}$ is called a *word* of *length* $n$ and is written $w = a_1a_2\cdots a_n$. The length of $w$ is denoted by $|w|$ and we denote by $|w|_a$ (resp. $|w|_b$) the number of occurrences of the letter $a$ (resp. $b$) in $w$.

A *right-infinite* (resp. *left-infinite*, *bi-infinite*) word over $\{a,b\}$ is a sequence indexed by $\mathbb{N}^+$ (resp. $\mathbb{Z}\backslash\mathbb{N}^+$, $\mathbb{Z}$) with values in $\{a,b\}$. For instance, a

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[1] Thanks are due to C. Reutenauer and J. Berstel, who introduced the authors to Christoffel words at the Centre des Recherches Mathématiques (Montréal), March, 2007.

left-infinite word is represented by $\boldsymbol{u} = \cdots a_{-2}a_{-1}a_0$ and a right-infinite word by $\boldsymbol{v} = a_1a_2a_3\cdots$, and their *concatenation* gives the bi-infinite word $\boldsymbol{uv} = \cdots a_{-2}a_{-1}a_0a_1a_2a_3\cdots$. Infinite words are typically typed in boldface.

If $v = a_1a_2\cdots$ is a finite or a right-infinite word, then its *reversal* $\widetilde{v}$ is the word $\cdots a_2a_1$. Similarly, if $\boldsymbol{u}$ is a left-infinite word, then its reversal is the right-infinite word $\widetilde{\boldsymbol{u}}$. We define the reversal of a bi-infinite word $\boldsymbol{s} = \cdots a_{-2}a_{-1}a_0a_1a_2\cdots$ by $\widetilde{\boldsymbol{s}} = \cdots a_2a_1a_0a_{-1}a_{-2}\cdots$. A finite word $w$ is a *palindrome* if $w = \widetilde{w}$.

A *factor* of a finite or infinite word $w$ is a finite word $v$ such that $w = uvu'$ for some words $u$, $u'$.

**Definition 2** (1) *Suppose $\boldsymbol{s}$ is a bi-infinite word on the alphabet $\{a, b\}$. We say that $\boldsymbol{s}$ satisfies the* **Markoff condition** *if for each factorization $\boldsymbol{s} = \widetilde{\boldsymbol{u}}xy\boldsymbol{v}$ with $\{x, y\} = \{a, b\}$, one has either $\boldsymbol{u} = \boldsymbol{v}$ or $\boldsymbol{u} = my\boldsymbol{u}'$ and $\boldsymbol{v} = mx\boldsymbol{v}'$ for some finite word $m$ (possibly empty) and right-infinite words $\boldsymbol{u}'$, $\boldsymbol{v}'$.*
(2) *A (finite) word $m$ is a* **Markoff word** *if there exists a bi-infinite word $\boldsymbol{s}$ satisfying the Markoff condition with a factorization of the form $\boldsymbol{s} = \widetilde{\boldsymbol{u}}y\widetilde{m}xymx\boldsymbol{v}$, where $\{x, y\} = \{a, b\}$.*

Note that a bi-infinite word $\boldsymbol{s}$ satisfies the Markoff condition if and only if its reversal $\widetilde{\boldsymbol{s}}$ does, and $\boldsymbol{s}$ does not satisfy the Markoff condition if and only if $\boldsymbol{s}$ or $\widetilde{\boldsymbol{s}}$ contains a factor of the form $a\widetilde{m}abmb$ for some finite word $m$.

Words $\boldsymbol{s}$ satisfying the Markoff condition fall into four classes: the periodic class; two aperiodic classes; and an ultimately periodic class. See Section 5. An example of each type appears below.

$$\cdots(aabab)(aabab)(aabab)(aabab)\cdots \tag{1}$$

$$\cdots abaababababaababababaababababa\cdots \tag{2}$$

$$\cdots abaabababababaababababaababababa\cdots \tag{3}$$

$$\cdots(baaa)(baaa)baab(aaab)(aaab)\cdots \tag{4}$$

## 3. The balanced property

Observe that the above examples of bi-infinite words are "balanced" in the following sense.

**Definition 3** *A finite or infinite word $w$ over $\{a, b\}$ is said to be* **balanced** *if for any two factors $u$, $v$ of $w$ with $|u| = |v|$, we have $\big||u|_a - |v|_a\big| \leq 1$ (or equivalently, $\big||u|_b - |v|_b\big| \leq 1$), i.e., the number of $a$'s (or $b$'s) in each of $u$ and $v$ differs by at most 1.*

This notion dates back to the seminal work of Morse and Hedlund [10]. More recently, Reutenauer proved the equivalence between the Markoff condition and the above balanced property:

**Proposition 4** [11, Theorem 3.1] *A bi-infinite word $\boldsymbol{s}$ satisfies the Markoff condition if and only if $\boldsymbol{s}$ is balanced.* □

In Section 5 we recount Reutenauer's Theorem 6.1 in [11], which gives a refinement of the balanced property and of the Markoff condition yielding the four classes illustrated by (1)–(4).

## 4. Central words

There exist several equivalent ways to define central words (see [7, Chapter 2]). Here we choose to use the following definition, as proved in [7, Proposition 2.2.34] using results from [4,5].

**Definition 5** *A word $w$ over $\{a, b\}$ is* **central** *if and only if $awb$ and $bwa$ are balanced.*

The following fact is especially pertinent.

**Lemma 6** *Any central word is a palindrome.*

**PROOF.** If $w$ is central, then $awb$ and $bwa$ are balanced by Definition 5. Arguing by contradiction, suppose $w$ is not a palindrome. There exist words $u$, $v$, $z$ and letters $\{x, y\} = \{a, b\}$ such that $w = uxv = zy\widetilde{u}$. But then

$$xwy = xuxvy = xzy\widetilde{u}y,$$

and the factors $xux$ and $y\widetilde{u}y$ contradict the balanced property of $xwy$. □

*Note.* Lemma 6 also appears under a different guise in [5, Lemma 7]. Also see Corollary 2.2.9 in [7]. The above proof is easily adapted to show directly that Markoff words are palindromes.

We are now ready to prove Theorem 1: *a word is a Markoff word if and only if it is a central word.*

**Proof of Theorem 1** Suppose $m$ is a Markoff word. Let $\boldsymbol{s}$ be a bi-infinite word satisfying the Markoff condition for which $y\widetilde{m}xymx$ is a factor, where $\{x, y\} = \{a, b\}$. The reversal of this factor, namely $x\widetilde{m}yxmy$, is a factor of $\widetilde{\boldsymbol{s}}$, which also satisfies the Markoff condition. Therefore, the words $amb$ and $bma$ are factors of bi-infinite words satisfying the Markoff condition, and hence are balanced by Proposition 4. Thus $m$ is central, by Definition 5.

Conversely, suppose $m$ is a central word. Then $m$ is a palindrome by Lemma 6, and moreover $amb = a\widetilde{m}b$ is balanced (Definition 5). Therefore the word $a\widetilde{m}bamb$ is also balanced and it can be viewed as a factor of some bi-infinite word satisfying the Markoff condition by Proposition 4—specifically, a bi-infinite word of the type represented in (1), with $amb$ repeated bi-infinitely. Thus $m$ is a Markoff word by Definition 2(2). $\square$

An immediate corollary is a new characterization of Christoffel words (defined in the next section).

**Corollary 7** *A word $m$ is a Markoff word if and only if $amb$ is a Christoffel word.*

**PROOF.** From [7, Chapter 2], a finite word $amb$ is Christoffel word if and only if $m$ is a central word, i.e., a Markoff word (by Theorem 1). $\square$

## 5. Christoffel words

This section describes four classes of words satisfying the Markoff condition and how they naturally coincide with four classes of balanced words.

If a bi-infinite word $\boldsymbol{s}$ satisfies the Markoff condition, then it falls into exactly one of the following classes.

Let $\{x, y\} = \{a, b\}$.

$(M_1)$ The lengths of the Markoff words $m$ occurring in $\boldsymbol{s}$ are bounded and $\boldsymbol{s}$ cannot be written as $\widetilde{\boldsymbol{u}}xy\boldsymbol{u}$ for some word $\boldsymbol{u}$.

$(M_2)$ The lengths of the Markoff words $m$ occurring in $\boldsymbol{s}$ are unbounded and $\boldsymbol{s}$ cannot be written as $\widetilde{\boldsymbol{u}}xy\boldsymbol{u}$ for some word $\boldsymbol{u}$.

$(M_3)$ There is exactly one $j \in \mathbb{Z}$ such that $\boldsymbol{s}$ has the factorization $\boldsymbol{s} = \widetilde{\boldsymbol{u}}s_js_{j+1}\boldsymbol{u}$ with $s_j \neq s_{j+1}$.

$(M_4)$ $\boldsymbol{s}$ is not of type $(M_1)$–$(M_3)$.

(Equivalently, $\boldsymbol{s}$ is in $(M_4)$ iff there exist at least two $i \in \mathbb{Z}$ such that $\boldsymbol{s} = \widetilde{\boldsymbol{u}}s_is_{i+1}\boldsymbol{u}$ with $s_i \neq s_{i+1}$.)

The four examples (1)–(4) in Section 2 correspond, respectively, to the classes $(M_1)$–$(M_4)$ above. We now turn to constructing words in each of the above classes. To achieve this, we present a geometric construction of Christoffel words, which allows for a description of balanced bi-infinite words.

Fix $p, q \in \mathbb{N}$, with $p$ and $q$ relatively prime. Let $\mathcal{P}$ denote the path in the integer lattice from $(0,0)$ to $(p,q)$ that satisfies: (i) $\mathcal{P}$ lies below the line segment $\mathcal{S}$ which begins at the origin and ends at $(p,q)$; and (ii) the region in the plane enclosed by $\mathcal{P}$ and $\mathcal{S}$ contains no other points of $\mathbb{Z} \times \mathbb{Z}$ besides those of $\mathcal{P}$.

Each step in $\mathcal{P}$ moves from a point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ to either $(x+1, y)$ or $(x, y+1)$, so we get a word $L(p,q)$ over the alphabet $\{a, b\}$ by encoding steps of the first type by the letter $a$ and steps of the second type by the letter $b$. See Figure 1.

The word $L(p,q)$ is called the **(lower) Christoffel word** of slope $\frac{q}{p}$. The *upper Christoffel words* are defined analogously.

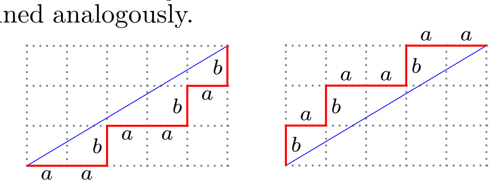


Fig. 1. The lower and upper Christoffel words of slope $\frac{3}{5}$ are $aabaabab$ and $babaabaa$, respectively.

For an introduction to the beautiful theory of Christoffel words, see [7, Chapter 2] or [1].

If the line segment $\mathcal{S}$ (as defined above) is replaced by a line $\ell$, then the construction produces balanced bi-infinite words. Moreover, all balanced bi-infinite words can be obtained by modifying this construction; they fall naturally into the following four classes determined by $\ell$.

$(B_1)$ $\ell(x) = \frac{q}{p}x$ is a line of rational slope $\frac{q}{p}$ (these are the periodic balanced words, see (1)).

$(B_2)$ $\ell$ is a line of irrational slope that does not meet any point of $\mathbb{Z} \times \mathbb{Z}$ (in (2), $\ell(x) = \frac{\pi}{4}x + e$).

$(B_3)$ $\ell(x) = \alpha x$ is a line of irrational slope meeting exactly one point of $\mathbb{Z} \times \mathbb{Z}$ (in (3), $\ell(x) = \frac{\pi}{4}x$).

$(B_4)$ The balanced words not of type $(B_1)$–$(B_3)$.

Balanced bi-infinite words of type $(B_4)$, represented in (4), are either of the form $\cdots xxyxx \cdots$ or $\cdots (ymx)(ymx)(ymy)(xmy)(xmy) \cdots$, where $\{x, y\} = \{a, b\}$ and $m$ is a Markoff word. Hence, it is possible to adapt the geometric construction above to construct this class of balanced words also. See Figure 2.

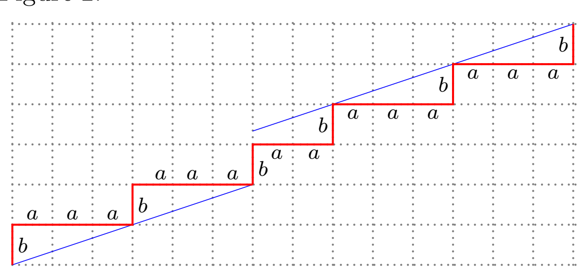


Fig. 2. Constructing example (4) $\cdots (baaa)baab(aaab) \cdots$.

As shown in [11], classes $(B_1)$–$(B_4)$ are derived from Morse and Hedlund's description of balanced bi-infinite words [10] (also see Heinis [6]).

**Proposition 8** [11, Theorem 6.1] *For $1 \le i \le 4$, one has the coincidences $(M_i) = (B_i)$.*

In closing, we mention that Markoff was interested in words over the alphabet $\{1,2\}$ that satisfy the Markoff condition. For these words, he studied the continued fraction quantities

$$\lambda_i(\boldsymbol{s}) = s_i + [0, s_{i+1}, \cdots] + [0, s_{i-1}, s_{i-2}, \cdots]$$

and $\Lambda(\boldsymbol{s}) = \sup_i \lambda_i(\boldsymbol{s})$. Reutenauer [11, Theorem 7.2] showed that classes $(M_1)$–$(M_4)$ correspond, respectively, to those $\boldsymbol{s}$ satisfying the Markoff condition with: $\Lambda(\boldsymbol{s}) < 3$; $\lambda_i(\boldsymbol{s}) < 3$ for all $i$ but $\Lambda(\boldsymbol{s}) = 3$; $\Lambda(\boldsymbol{s}) = 3 = \lambda_i(\boldsymbol{s})$ for a unique $i \in \mathbb{Z}$; $\Lambda(\boldsymbol{s}) = 3 = \lambda_i(\boldsymbol{s})$ for at least two $i \in \mathbb{Z}$.

The set $\{\Lambda(\boldsymbol{s}) \mid \boldsymbol{s} \text{ is a bi-infinite word over } \mathbb{N}^+\}$, with none of the conditions on $\boldsymbol{s}$ originally imposed by Markoff, has become known as the **Markoff spectrum**. Results and open questions concerning the Markoff spectrum may be found in [2].

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